Onset of the convection in a supercritical fluid

H. Meyer
Department of Physics, Duke University, Durham, North Carolina 27708-0305, USA
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A model is proposed that leads to the scaled relation $t_p/\tau_D = F_{tp}(\mathrm{Ra}-\mathrm{Ra}_c)$ for the development of convection in a pure fluid in a Rayleigh-Bénard cell after the start of the heat current at $t=0$. Here t_p is the time of the first maximum of the temperature drop $\Delta T(t)$ across the fluid layer, the signature of rapidly growing convection, τ_D is the diffusion relaxation time, and Ra_c is the critical Rayleigh number. Such a relation was first obtained empirically from experimental data. Because of the unknown perturbations in the cell that lead to convection development beyond the point of the fluid instability, the model determines t_p/τ_D within a multiplicative factor $\Psi\sqrt{\mathrm{Ra}_c(\mathrm{HBL})}$, the only fit parameter product. Here $\mathrm{Ra}_c(\mathrm{HBL})$, of the order 10^3, is the critical Rayleigh number of the hot boundary layer and Ψ is a fit parameter. There is then good agreement over more than four decades of $\mathrm{Ra}-\mathrm{Ra}_c$ between the model and the experiments on supercritical ^{3}He at various heat currents and temperatures. The value of the parameter Ψ, which phenomenologically represents the effectiveness of the perturbations, is discussed in connection with predictions by El Khouri and Carlès of the fluid instability onset time.

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Consider a pure fluid in a Rayleigh-Bénard cell after a constant heat current q across the sample has been turned on at the time $t=0$. The purpose of this paper is to address the question of how the convection develops as a function of t beyond the instability point of the fluid.

Recently experimental studies of convection in a Rayleigh-Bénard (RB) cell were published for a pure supercritical fluid, ^{3}He, at several temperatures along its critical isochore [1–3]. In the experiments, the temperature drop $\Delta T(t)$ across the fluid layer was measured as a function of t after the start of the heat current q. The measurements included both the *steady state* temperature drop in the convective regime, $\Delta T = \Delta T(q,\epsilon)$, with $\epsilon \equiv (T-T_c)/T_c$ and $T_c = 3.318$ K the critical temperature, and also the *transient features*. The latter were the time t_p of the first peak in $\Delta T(t)$ after the start of the heat current, the period t_{osc} of the damped oscillations in the $\Delta T(t)$ profile, and τ_{tail}, the asymptotic relaxation time to the steady state. It was found empirically that these characteristic times at the various values of ϵ and q, scaled by the thermal diffusion relaxation time $\tau_D(\epsilon)$, collapse each on its respective "universal" curve when plotted versus the Rayleigh number. For instance, it was found that

$$t_p/\tau_D = F_{tp}(\mathrm{Ra}_{corr} - \mathrm{Ra}_c), \tag{1}$$

where F_{tp} is a function of the quantity in parentheses, and similarly there were functions F_{osc} and F_{tail} for t_{osc} and τ_{tail}. In the limit of $\gamma = C_P/C_V \gg 1$, the diffusion relaxation time is given by $\tau_D = L^2/4D_T$ [4]. Here D_T is the thermal diffusivity, L is the layer height, and C_P and C_V are the specific heats at constant pressure and volume. Also Ra_{corr} is the Rayleigh number corrected for the adiabatic temperature gradient, as defined in Refs. [1,2], and $\mathrm{Ra}_c = 1708$ is the critical Rayleigh number for a RB cell with a large aspect ratio. Comparison of numerical simulations with experiments, both for the steady state and for the transient results, was described in Ref. [3]. In Ref. [5] the discrepancies for the time t_p between the simulations and experiments were studied by introducing perturbations into the simulations. The purpose of these was to imitate the noise or perturbations, present in the physical system, which are needed to develop the instability. In the simulations, various amplitudes of a spatially periodic temperature perturbation, imposed along one of the parallel plates of the RB cell, were tested. A small amplitude of ≈ 0.5 μK produced a development of convection in a time comparable to that in the physical system.

In this paper a simple model is presented which justifies the scaling relation (1), and where the predictions are consistent with this empirical representation. Our model is based on the assumption that a portion of the fluid first becomes unstable at the time $t_{HBL\ instab}$ when the Rayleigh number of the "hot" boundary layer (HBL) at the bottom boundary reaches its critical value. The scenario is then as follows. The fluid is kept at a constant average density in a Rayleigh-Bénard cell of large aspect ratio and where the critical Rayleigh number of the fluid between two solid boundaries is $\mathrm{Ra}_c = 1708$. The high thermal conductivity of the plates keeps their respective temperatures homogeneous. Upon the start of the heat current q at $t=0$, boundary layers form at both plates. Through the "piston effect" [6,7], the temperature of the bulk fluid is homogenized after a characteristic piston time $t_1 = L^2/[D_T(\gamma-1)^2]$.

The temperature drop $\Delta T(t)$ across the total fluid layer L is the sum of the drops $\delta T(t)_{HBL}$ and $\delta T(t)_{CBL}$ across the "hot" and the "cold" (CBL) boundary layers at the hot and the cold surfaces of the plates. This is shown in Fig. 1 by plots from numerical calculations first presented in Fig. 7a of Ref. [8] for a thermal conductivity cell with the heat flowing downward, or conversely for a RB cell in the absence of earth's gravity. These were extended by Zhong [9] to a cell height $L = 0.106$ cm used in the convection experiments of Ref. [1]. From Fig. 1 it is seen that the temperature drops are comparable, with the hot layer drop $\delta T(t)_{HBL}(t)$ the larger one by $\approx 20\%$. The ratio $\delta T(t)_{HBL}/\delta T(t)_{CBL}$ has been checked in several plots similar to Fig. 1 at different values

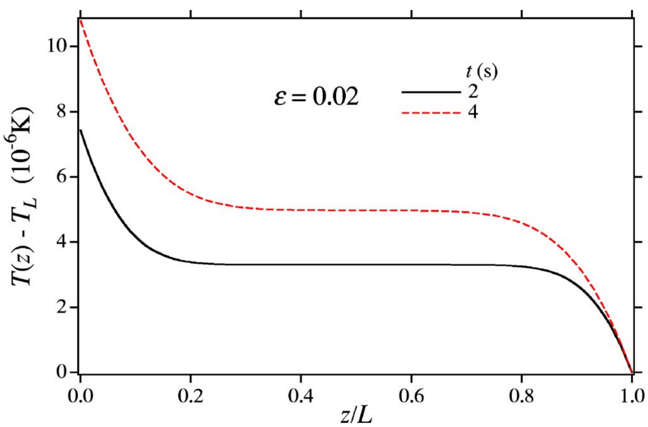

FIG. 1. (Color online) The vertical temperature distribution in a thermal conductivity cell with layer height $L=0.106$ cm at $t=2$ and 4 s, and for $\epsilon = 0.02$, and $q=0.1\ \mu\mathrm{W/cm^2}$, as obtained from the numerical calculations presented in Ref. [8].

of ϵ and was found to vary from ≈ 1 at $\epsilon=0.01$ to ≈ 2 at $\epsilon=0.14$. There was no definite trend of this ratio with increasing q at a given ϵ. (See also Ref. [10].) We then set

$$\delta T(t)_{HBL} = (1+\Phi)\Delta T(t)/2, \qquad (2)$$

and use an average value of $\Phi = 0.2 \pm 0.1$ taken for curves calculated over the range of ϵ between 0.01 and 0.14. Because Φ increases more rapidly for ϵ above ≈ 0.14, the Φ values at higher ϵ are not used in the analysis and the discussion below. The thickness of both boundary layers is seen from Fig. 1 to be almost the same, and approximated (see [7]) by $l_{HBL} \approx l_{CBL} \approx (D_T t)^{1/2}$. We set

$$l_{HBL} = A(D_T t)^{1/2}, \qquad (3)$$

where A is obtained by defining l_{HBL} as the distance determined by 99% of the temperature drop from the wall to the flat temperature profile in the central part of the fluid layer. An average value of $A=3.5\pm0.2$ is then obtained from calculated curves at various values of ϵ between 0.01 and 0.14, similar to those in Fig. 1.

Initially the fluid is then in the nonconvecting mode, and the temperature drop across the fluid layer is given by [11]

$$\Delta T(t) = \frac{q}{\lambda}\sqrt{\frac{D_T t}{\pi}}\left[4 - \left(\frac{\pi t_1}{t}\right)^{1/2}\right], \quad t \gg t_1, \qquad (4)$$

where λ is the fluid thermal conductivity. For large enough heat currents, l_{HBL} will reach a critical size, and is the first boundary layer to become unstable at $t=t_{HBL\ instab}$. This is because the product $l_{HBL}^3 \times \delta T_{HBL}$, which is proportional to the local Rayleigh number, is larger than that for the cold boundary layer, as shown from the example at $\epsilon=0.02$ at $t=2$ and 4 s in Fig. 1. The instability condition for this layer is then [12,13]

$$\mathrm{Ra(HBL)} = \frac{g l_{HBL}^3 \alpha_P}{D_T \nu}\left(\delta T_{HBL} - \frac{g T \alpha_P l_{HBL}}{C_P}\right) > \mathrm{Ra}_c(\mathrm{HBL}). \qquad (5)$$

The second term in parentheses is the contribution of the adiabatic temperature gradient. Here the local critical Rayleigh number $\mathrm{Ra}_c(\mathrm{HBL})$ corresponds approximately to the condition "one rigid and one free bounding surface," with the predicted value 1100 [14]. This numerical value, obtained for idealized conditions of two parallel flat surfaces, might not strictly apply to the hot boundary layer, where the upper boundary with the bulk fluid is not "free" nor sharply defined. Its actual value might be of the order of 10^3, and this implication is discussed in the result analysis below. Also α_P is the coefficient of isobaric thermal expansion, g the gravitational acceleration, and ν the kinematic viscosity. In a recent paper, Accary *et al.* [15] have studied numerically in a two-dimensional (2D) approximation the hydrodynamic stability of a supercritical fluid layer in a RB cell after the bottom plate temperature was raised quickly by ΔT, with the temperature of the top plate being kept constant. They found that it was the hot boundary layer (labeled as the "active" one) that became unstable first, which is consistent with the assumption in the experimental conditions described above, where a constant heat current is imposed at the bottom plate.

It is the purpose of this paper to estimate the time when this instability has developed significantly into convection, and which is represented by the time t_p, the time of the first peak in $\Delta T(t)$ when convection plumes are reaching the top of the fluid layer, and to relate it to the Rayleigh number of the whole fluid layer. Clearly, a quantitative estimation of t_p cannot be made, because the nature and amplitude of the perturbations that lead to the development of convection for $t > t_{HBL\ instab}$ are not known. The following *ad hoc* postulate, to be discussed later in conjunction with the results, is now made:

$$t_p = \Psi \times t_{HBL\ instab}, \qquad (6)$$

where Ψ is a parameter representing phenomenologically the effects from perturbations such as imperfections in the RB cell, wall effects, etc. A further postulate is that Ψ is the same for all the experiments done in the RB cell filled with supercritical ^{3}He over a temperature range of about 0.6 K. When the perturbation amplitude is negligible, $\Psi \gg 1$, and the convection takes a long time to develop fully. This is shown by numerical 2D simulations [5] for the example of ^{3}He at $\epsilon=0.2$ and $q=2.16\times10^{-7}$ W/cm^2 with cell aspect ratio $\Gamma=8$ and $L=0.106$ cm, where convection takes 90 s to become well developed when only the inherent noise of the simulation code is present. In unpublished 2D numerical simulations by Accary [16] with a different code but with the same fluid parameters as in the experiments, and for $\Gamma=8$ and $L=0.106$ cm, the convection develops significantly after 120 s. Finally, in simulations by Amiroudine [17], with the same fluid parameters and cell height, but for an aspect ratio of $\Gamma=2$, the peak of $\Delta T(t)$ occurred at $t_p=55$ s. These times are larger than in the experimental system with $t_p=30$ s. They are to be contrasted with $t_{instab}=7$ s for the whole fluid layer with the same parameters as in the experiments, calculated from first principles by El Khouri and Carlès [18]. On the other hand, when the perturbations are very strong, one expects $\Psi \to 1$, as t_p decreases toward $t_{HBL\ instab}$.

Under the experimental conditions for the data reported in Ref. [3], numerical estimates show that for times near and above the HBL instability and for the values of ϵ used, both second terms in brackets of Eq. (4) and in parentheses of Eq. (5) are small compared to the first ones, and henceforth are neglected, which simplifies the calculations. Combining Eqs. (1)–(6), one has at the time of instability of the hot boundary layer

$$\mathrm{Ra}_c(\mathrm{HBL}) = \frac{2A^3(1+\Phi)gD_T\alpha_P}{\Psi^2\pi^{1/2}\nu\lambda} \times t_p^2 \times q. \tag{7}$$

It is convenient to express q in terms of the Nusselt number, which is a function of the Rayleigh number of the fluid layer. Once the instability has led to a fully developed convection and steady state has been reached, the connection between the heat current q and the temperature drop ΔT can be obtained via the relation

$$\mathrm{Nu}_{corr} = \mathrm{Nu}_{corr}(\mathrm{Ra}_{corr}) \tag{8}$$

where Nu_{corr} is the Nusselt number corrected for the adiabatic temperature gradient [1,2],

$$\mathrm{Nu}_{corr} = \frac{qL/\lambda - \Delta T_{ad}}{\Delta T - \Delta T_{ad}}, \tag{9}$$

and

$$\mathrm{Ra}_{corr} = \frac{gL^3\alpha_P}{D_T\nu}(\Delta T - \Delta T_{ad}). \tag{10}$$

Here $qL/\lambda = \Delta T_{cond}$ is the temperature drop across the fluid layer for the conducting (nonconvecting) fluid with the same heat current q, while ΔT is the measured steady state temperature drop in the convecting state. Also $\Delta T_{ad} = gTL\alpha_P/C_P$, the contribution of the adiabatic temperature gradient over the sample height. In Eq. (8) the dependence of Nu on the Prandtl number has been ignored, as it was found to be small from the experimental data analysis [1].

For the steady state $\Delta T(q,\epsilon)$ data we will use the results from Ref. [1] in Fig. 6, which are the same as in Fig. 3(a) of Ref. [3], namely, where $\mathrm{Nu}_{corr}(q,\epsilon)$ is expressed by the convection heat current j_{corr}, first introduced in Ref. [19] and used in Refs. [1,3]:

$$j_{corr} = (\mathrm{Nu}_{corr} - 1)(1 + a^*_{corr}) \tag{11}$$

with

$$a^*_{corr} = (\mathrm{Ra}_{corr} - \mathrm{Ra}_c)/\mathrm{Ra}_c, \tag{12}$$

and hence

$$\mathrm{Nu}_{corr} = \frac{1 + a^*_{corr} + j_{corr}}{1 + a^*_{corr}} = \frac{qL/\lambda - \Delta T_{ad}}{\Delta T - \Delta T_{ad}}. \tag{13}$$

Because qL/λ is larger than ΔT, and we use values so that $qL/\lambda \gg \Delta T_{ad}$, then

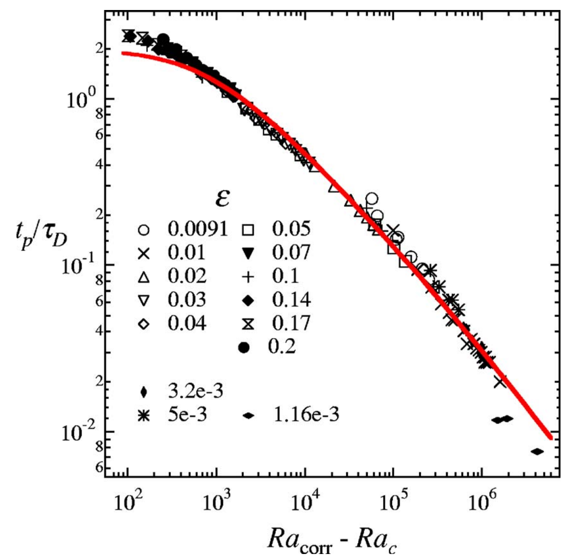

FIG. 2. (Color online) The time of the first peak, t_p, scaled by the diffusive relaxation time τ_D, vs $\mathrm{Ra}_{corr} - \mathrm{Ra}_c$. The symbols represent the experimental data at various values of ϵ as per Ref. [3] and the solid line the predictions as expressed by Eq. (16) with the fitting parameter $\Psi = 4.5$.

$$q = (\Delta T - \Delta T_{ad})\frac{\lambda}{L}\frac{(1 + a^*_{corr} + j_{corr})}{(1 + a^*_{corr})}. \tag{14}$$

A two-term expression could satisfactorily fit the experimental data shown in Fig. 3(b) of Ref. [3] and is given by

$$j_{corr} = 1.3a^*_{corr} + 0.287(a^*_{corr})^{1.45}. \tag{15}$$

Finally, combining Eqs. (7), (14), and (15), and using Eq. (12) and the diffusion relaxation time $\tau_D = L^2/4D_T$ for the condition $\gamma \gg 1$, one obtains

$$\frac{t_p}{\tau_D} = \Psi\sqrt{\mathrm{Ra}_c(\mathrm{HBL})} \times \frac{B}{(1 + 2.3a^*_{corr} + 0.287a^{*1.45}_{corr})^{1/2}}, \tag{16}$$

where

$$B = \sqrt{\frac{8\pi^{1/2}}{Ra_c A^3(1+\Phi)}}. \tag{17}$$

Hence effectively the model predicts t_p/τ_D to be a function of $\mathrm{Ra}_{corr} - \mathrm{Ra}_c$, in agreement with the empirical findings. The result of Eq. (16) is shown in Fig. 2 where the fit parameter is $\Psi\sqrt{\mathrm{Ra}_c(\mathrm{HBL})} = 166 \pm 25$, given the values in Eq. (17) of $\mathrm{Ra}_c = 1708$, $A = 3.5$ for the amplitude of the hot boundary layer width, and $\Phi = 0.2$. By using tentatively the value $\mathrm{Ra}_c(\mathrm{HBL}) = 1100$, a first reasonable guess as mentioned earlier, one obtains $\Psi = 4.7$. Because the $\mathrm{Ra}_c(\mathrm{HBL})$ value might well be not better known than its order of magnitude of 10^3, this in turn leads to uncertainty in Ψ, though their product is quite well determined. The calculated curve extends over the range $10^2 \leq (\mathrm{Ra}_{corr} - \mathrm{Ra}_c) \leq 7 \times 10^6$ [20]. The fit is satisfactory for $(\mathrm{Ra}_{corr} - \mathrm{Ra}_c) > 5 \times 10^{-2}$, but there is a systematic deviation from the experimental data curve at lower values of $Ra_{corr}-Ra_c$. We note that the data for these lower values are predominantly associated with higher values of ϵ where D_T has become large, low values of q, and where the HBL and CBL are no longer well separated near $t_{HBL\ instab}$. At that time the instability becomes dominated by global modes in the whole cell (scenarios 1 and 2 in Ref. [18]) instead of the HBL [10]. The approximations made in this simple model are then no longer justified, but it should be noted that the scaling according to Eq. (1) is still followed by the collection of experimental data. Hence $t_p \approx 4.7 \times t_{HBL\ instab}$. For the example of ^{3}He along the critical isochore at $\epsilon=0.1$ for $q=6.22\times 10^{-8}$ W/cm^2, the experimental value is $t_p \approx 43$ s. Then one calculates $t_{LBH\ instab}=t_p/\Psi \approx 9.1$ s. Within the uncertainty of Ψ, this is consistent with the instability time predicted for the fluid layer from first principles (Ref. [18]) of $t_{instab}=12$ s. One further test example is at $\epsilon=0.02$ and $q=5.1\times 10^{-8}$ W/cm^2. Here the predicted $t_{instab}=14$ s [18]. This compares with $t_{HBL\ instab} \approx 8.5$ s, obtained from the experimental $t_p=40$ s.

The satisfactory agreement of the scaled curve obtained from the simple model and the experimental data—except where indicated—seems to justify the postulate made in Eq. (6) relating the time t_p and $t_{HBL\ instab}$. The fit parameter Ψ can be interpreted as a phenomenological measure of the effectiveness of the perturbations in the fluid cell used in the experiments: The larger the value of Ψ, the smaller their effectiveness in developing convection. Of course, one expects Ψ to be dependent on the cell configuration, aspect ratio, roughness, and alignment of the horizontal plates and vertical boundaries, etc.

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[20] The reader will have noticed that data points for $\epsilon=0.01$ in Fig. 2 do not extend to as high values of $Ra_{corr}-Ra_c$ as those in Fig. 3(a) of Ref. [3]. This is because for these high values, the first peak with time t_p is no longer recorded experimentally. After the sharp initial rise of $\Delta T(t)$, this peak is replaced by a smooth increase to the asymptotic steady state value, and no damped oscillations are observed either. This is an instrumental effect where the time constant of the temperature recording system is larger than the width of the very narrow peak and of the damped oscillations. As a result these sharp features are smoothed out.